積分難題解析

大學微積分---代換型或分部積分型

http://stanley-math.webnode.tw　史丹利的數學世界

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重點精研

◎主要公式

※一般公式

1. $\int k dx = kx + C$

2. $\int ax^n dx = a\dfrac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$

3. $\int \dfrac{k}{x}dx = k\ln|x| + C$

4. $\int u dv = uv - \int v du$

5. $\int kf(x)dx = k\int f(x)dx$

6. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

※Gamma 函數

1. $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ ， $x > 0$

2. $\Gamma(x+1) = x\Gamma(x)$

3. $\Gamma(n+1) = n!$

4. $\Gamma(\dfrac{1}{2}) = \sqrt{\pi} = \int_0^\infty \dfrac{e^{-t}}{\sqrt{t}}dt$

※Beta 函數

1. $\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$ ， $p > 0, q > 0$

2. $\beta(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

$Evaluate \int \ln(x^2+2x+2)\,dx$ (72 中央大學)

[sol] $x^2+2x+2=(x+1)^2+1$

令$u=x+1 \quad du=dx$

原式 $=\int \ln(u^2+1)\,du$ 令$u=tan\theta \;\; du=sec^2\theta d\theta \quad \theta=\tan^{-1}u=\tan^{-1}(x+1)$

$\rightarrow \int \ln(sec^2\theta)\,sec^2\theta d\theta = \; 2\int \ln(sec\theta)sec^2\theta d\theta$

令$I=\int \ln(sec\theta)sec^2\theta d\theta = \int \ln(sec\theta)dtan\theta$

分部積分 $\rightarrow tan\theta \ln(sec\theta) - \int tan\theta\, dln(sec\theta)$

$= tan\theta \ln(sec\theta) - \int tan^2\theta\, d\vartheta = tan\theta \ln(sec\theta) - \int sec^2\theta - 1d\vartheta$

$= tan\theta \ln(sec\theta) - tan\theta + \theta = \frac{1}{2}tan\theta \ln(sec^2\theta) - tan\theta + \theta$

又$\theta=\tan^{-1}(x+1) \rightarrow I=\frac{1}{2}(x+1)\ln((x+1)^2+1)-(x+1)+\tan^{-1}(x+1)$

又所求積分為 $2I=(x+1)\ln(x^2+2x+2)-2(x+1)+2\tan^{-1}(x+1)+c$

<註記>可以看到$-2(x+1)$常數部分的-2亦可與c合併形成c_2

$(x+1)\ln(x^2+2x+2)-2x+2\tan^{-1}(x+1)+c_2$

$Evaluate \int x\ln\left(x+\sqrt{1+x^2}\right)dx$ (交通大學)

[sol] 令$x=tan\theta \;\; dx=sec^2\theta d\theta \quad \theta=\tan^{-1}x$

$\rightarrow \int tan\theta sec^2\theta \ln(tan\theta+sec\theta)\,d\theta = \int \ln(tan\theta+sec\theta)d\frac{tan^2\theta}{2}$

$\rightarrow \frac{1}{2}[tan^2\theta \ln(tan\theta+sec\theta) - \int tan^2\theta\, sec\theta d\theta]$

又$\int tan^2\theta\, sec\theta d\theta = \int sec^3\theta - sec\theta d\theta = \int sec\theta dtan\theta - \ln(tan\theta+sec\theta)$

$=\frac{1}{2}(sec\theta tan\theta - \ln(sec\theta+tan\theta))$

原積分 $=\frac{1}{2}tan^2\theta \ln(tan\theta+sec\theta) - \frac{1}{4}sec\theta tan\theta + \frac{1}{4}\ln(sec\theta+tan\theta)$

又$\theta=\tan^{-1}x \;\rightarrow \frac{x^2}{2}\ln\left(x+\sqrt{1+x^2}\right)-\frac{x}{4}\sqrt{1+x^2}+\frac{1}{4}\ln\left(x+\sqrt{1+x^2}\right)+c$

$Evaluate \int \frac{1-tanx}{1+tanx}dx$

[sol]　原式 $= \int \frac{cosx - sinx}{cosx + sinx}dx = \int \frac{d(cosx + sinx)}{cosx + sinx}$

$= \ln(cosx + sinx) + c_1$

又和差化積公式$sinA + sinB = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

又$cosx + sinx = \sin\left(\frac{\pi}{2} - x\right) + \sin(x) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

$\rightarrow \ln\left[\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right] + c_1 = \ln\left[\sin\left(x + \frac{\pi}{4}\right)\right] + c_2$

< 另解 >　　原式 $= \int \frac{1 - tan\theta}{\tan\left(\frac{\pi}{4}\right) + tan\theta}d\theta = \int \frac{1 - \frac{\sin(\theta)}{\cos(\theta)}}{\frac{\sin(\frac{\pi}{4})}{\cos(\frac{\pi}{4})} + \frac{\sin(\theta)}{\cos(\theta)}}d\theta$

$= \int \frac{\cos(\theta)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)\sin(\theta)}{\sin\left(\frac{\pi}{4}\right)\cos(\theta) + \cos\left(\frac{\pi}{4}\right)\sin(\theta)}d\theta$，又 $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$

$= \int \frac{\cos(\theta)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\sin(\theta)}{\sin\left(\frac{\pi}{4}\right)\cos(\theta) + \cos\left(\frac{\pi}{4}\right)\sin(\theta)}d\theta$

又正餘弦和角公式分別為:

1. $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
2. $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$\therefore \int \frac{\cos(\theta)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\sin(\theta)}{\sin\left(\frac{\pi}{4}\right)\cos(\theta) + \cos\left(\frac{\pi}{4}\right)\sin(\theta)}d\theta = \int \frac{\cos(\theta + \frac{\pi}{4})}{\sin(\theta + \frac{\pi}{4})}d\theta$

$= \int \cot(\theta + \frac{\pi}{4})\,d\theta = \ln\left[\sin\left(\theta + \frac{\pi}{4}\right)\right] + C$

$Evaluate \int_0^{\frac{\pi}{2}} \frac{x}{tanx} dx$

[sol] $\int_0^{\frac{\pi}{2}} \frac{x}{tanx} dx$

$= \int_0^{\frac{\pi}{2}} xd(ln|sinx|) = xln|sinx| \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} ln|sinx|dx$

$= -\int_0^{\frac{\pi}{2}} ln|sinx|dx$

又令$x = \frac{\pi}{2} - v \quad dx = -dv$

$\therefore \int_0^{\frac{\pi}{2}} ln|sinx|dx = \int_0^{\frac{\pi}{2}} ln|cosv|dv$

故 $2\int_0^{\frac{\pi}{2}} ln|sinx|dx = \int_0^{\frac{\pi}{2}} ln\left|\frac{sin2x}{2}\right| dx$

又 $\int_0^{\frac{\pi}{2}} ln\left|\frac{sin2x}{2}\right| dx = \int_0^{\frac{\pi}{2}} ln|sin2x| - ln2dx$

$= \int_0^{\frac{\pi}{2}} ln|sin2x|dx - \frac{\pi}{2} ln2$

又令$u = 2x \quad dx = \frac{1}{2} du$

故等於$\frac{1}{2}\int_0^{\pi} ln|sinu|\, du - \frac{\pi}{2} ln2 = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} ln|sinu|\, du + \int_{\frac{\pi}{2}}^{\pi} ln|sinu|du\right] - \frac{\pi}{2} ln2$

又令$u = \pi - \omega \quad du = -d\omega$

$= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} ln|sin\omega|\, d\omega + \int_0^{\frac{\pi}{2}} ln|sin\omega|\, d\omega\right] - \frac{\pi}{2} ln2$

$\therefore 2\int_0^{\frac{\pi}{2}} ln|sinx|dx = \int_0^{\frac{\pi}{2}} ln|sinx|dx - \frac{\pi}{2} ln2$

故 $\int_0^{\frac{\pi}{2}} ln|sinx|dx = -\frac{\pi}{2} ln2$

又所求 $\int_0^{\frac{\pi}{2}} \frac{x}{tanx} dx = -\int_0^{\frac{\pi}{2}} ln|sinx|dx = \frac{\pi}{2} ln2$

$Evaluate \int \frac{ln(x+1)-ln(x)}{x(x+1)}dx$

[sol] 原式 $= \int \frac{1}{x^2} \times \frac{\ln\left(1+\frac{1}{x}\right)}{\left(1+\frac{1}{x}\right)} dx$

令$u = 1 + \frac{1}{x}$ 　 $du = \frac{1}{x^2} dx$

$\rightarrow \int \frac{\ln(u)}{u} du = \int \ln(u)\, dln(u) = \frac{\ln(u)^2}{2} + c$

$= \frac{[\ln(1+\frac{1}{x})]^2}{2} + c = \frac{[\ln(x+1) - \ln(x)]^2}{2} + c$

$Evaluate \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

[sol] 令$x = v^2$ $dx = 2vdv$ $v = \sqrt{x}$

$\rightarrow \int \frac{e^v}{v} \times 2vdv = \int 2e^v dv$

$= 2e^v + c$

又$v = \sqrt{x}$

原積分 $= 2e^{\sqrt{x}} + c$

$Evaluate \int_0^{\frac{\pi}{2}} \frac{sin^m x}{sin^m x + cos^m x} dx$

[sol] 令$x = \frac{\pi}{2} - y$ $dx = -dy$

$\rightarrow \int_0^{\frac{\pi}{2}} \frac{cos^m y}{sin^m y + cos^m y} dy = \int_0^{\frac{\pi}{2}} \frac{cos^m x}{sin^m x + cos^m x} dx$

此時的$x.y$均為啞變數，所謂啞變數及乃變數並不影響積分之值，通常一元的定積分基本上多為啞變數

又 $2\int_0^{\frac{\pi}{2}} \frac{sin^m x}{sin^m x + cos^m x} dx = \int_0^{\frac{\pi}{2}} \frac{sin^m x}{sin^m x + cos^m x} dx + \int_0^{\frac{\pi}{2}} \frac{cos^m x}{sin^m x + cos^m x} dx$

$= \int_0^{\frac{\pi}{2}} \frac{sin^m x + cos^m x}{sin^m x + cos^m x} dx = \frac{\pi}{2}$

$\therefore$ 原積分為$\frac{\pi}{4}$

$Evaluate \int_0^{\frac{\pi}{2}} \frac{1}{1+tan^{\sqrt{3}}x} dx$

[sol]　括分 → $\int_0^{\frac{\pi}{2}} \frac{cos^{\sqrt{3}}x}{cos^{\sqrt{3}}x + sin^{\sqrt{3}}x} dx$

滿足上題，且$m = \sqrt{3}$　故為$\frac{\pi}{4}$

$Evaluate \int_0^{1} \frac{\ln(1+x)}{1+x^2} dx$

[sol] 令$x = tan\theta \quad dx = sec^2\theta d\theta \quad \theta = tan^{-1} x$

$\rightarrow \int_0^{\frac{\pi}{4}} \ln(1+tan\theta)\, d\theta = \int_0^{\frac{\pi}{4}} \ln(cos\theta + sin\theta)\, d\theta - \int_0^{\frac{\pi}{4}} \ln(cos\theta)\, d\theta$

又$\int_0^{\frac{\pi}{4}} \ln(cos\theta + sin\theta)\, d\theta = \int_0^{\frac{\pi}{4}} \ln\left(\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)\right) d\theta$

$= \frac{\pi}{8} ln2 + \int_0^{\frac{\pi}{4}} \ln\left(\sin\left(\theta + \frac{\pi}{4}\right)\right) d\theta$

令 $\theta = \frac{\pi}{4} - y \quad d\theta = -dy$

$\rightarrow \int_0^{\frac{\pi}{4}} \ln\left(\sin\left(\frac{\pi}{2} - y\right)\right) dy = \int_0^{\frac{\pi}{4}} \ln(\cos y)\, dy = \int_0^{\frac{\pi}{4}} \ln(\cos\theta)\, d\theta$

∴ 原積分$\frac{\pi}{8} ln2 + \int_0^{\frac{\pi}{4}} \ln(\cos\theta)\, d\theta - \int_0^{\frac{\pi}{4}} \ln(\cos\theta)\, d\theta = \frac{\pi}{8} ln2$

$Evaluate \int_0^{\infty} \frac{\ln(x)}{1+x^2} dx$

[sol] 令$x = tan\theta \quad dx = sec^2\theta d\theta \quad \theta = tan^{-1} x$

$\rightarrow \int_0^{\frac{\pi}{2}} \ln(tan\theta)\, d\theta = \int_0^{\frac{\pi}{2}} \ln(sin\theta)\, d\theta - \int_0^{\frac{\pi}{2}} \ln(cos\theta)\, d\theta$

其中$\int_0^{\frac{\pi}{2}} \ln(sin\theta)\, d\theta = \int_0^{\frac{\pi}{2}} \ln(cos\theta)\, d\theta$

故積分為 0

Evaluate $\int_0^{2\pi} \frac{dx}{1+\tan^4 x}$

[sol] 原式 → $\int_0^{\pi} \frac{dx}{1+\tan^4 x} + \int_{\pi}^{2\pi} \frac{dx}{1+\tan^4 x}$

又令$x = \pi - y \quad dx = -dy$

$$\therefore \int_{\pi}^{2\pi} \frac{dx}{1+tan^4 x} = \int_{-\pi}^{0} \frac{dy}{1+tan^4 y}$$

再令$y = -\theta \quad dy = -d\theta$ 其中θ為啞變數

$$\therefore \int_{-\pi}^{0} \frac{dy}{1+tan^4 y} = \int_0^{\pi} \frac{d\theta}{1+tan^4\theta} = \int_0^{\pi} \frac{dx}{1+tan^4 x}$$

故 $\int_0^{2\pi} \frac{dx}{1+tan^4 x} = 2\int_0^{\pi} \frac{dx}{1+tan^4 x} = 2I$

取$x = \frac{v}{2} \quad dx = \frac{dv}{2} \quad v = 2x$

$I = \frac{1}{2}\int_0^{2\pi} \frac{dv}{1+tan^4(\frac{v}{2})}$ 又正切半角公式$tan\frac{x}{2} = \sqrt{\frac{1-cosx}{1+cosx}}$

$$I = \frac{1}{2}\int_0^{2\pi} \frac{dx}{1+\left(\frac{1-cosx}{1+cosx}\right)^2} = \frac{1}{2}\left[\int_0^{2\pi} \frac{1+cos^2 x}{2(1+cos^2 x)}dx + \int_0^{2\pi} \frac{2cosx}{2+2cos^2 x}dx\right]$$

其中 $\int_0^{2\pi} \frac{2cosx}{2+2cos^2 x}dx$ 令$x = \pi - y \ dx = -dy$

$= \int_{-\pi}^{\pi} \frac{-2cosy}{2+2cos^2 y}dy$ 其中$\frac{-2cosy}{2+2cos^2 y}$為奇函數 故積分為 0

$\therefore I = \frac{\pi}{2}$ 又所求$2I = \pi$

Evaluate $\int_0^1 x(\tan^{-1} x)^2\, dx$

令$x = tan\theta \quad dx = sec^2\theta d\theta$ 其中$\theta = \tan^{-1}\sqrt{x}$

$= \int_0^{\frac{\pi}{4}} \theta^2 tan\theta sec^2\theta d\theta = \int_0^{\frac{\pi}{4}} \theta^2\, d\frac{tan^2\theta}{2}$

$= \theta^2\frac{tan^2\theta}{2} - \int_0^{\frac{\pi}{4}} \frac{tan^2\theta}{2} d\theta^2$

$= \theta^2\frac{tan^2\theta}{2} - \int_0^{\frac{\pi}{4}} \theta\, tan^2\theta d\theta = \theta^2\frac{tan^2\theta}{2} - \int_0^{\frac{\pi}{4}} \theta\,(sec^2\theta - 1)d\theta$

$$= \theta^2\frac{\tan^2\theta}{2} + \frac{\theta^2}{2} - \int_0^{\frac{\pi}{4}} \theta \sec^2\theta d\theta = \theta^2\frac{\sec^2\theta}{2} - \int_0^{\frac{\pi}{4}} \theta d\tan\theta$$

$$= \theta^2\frac{\sec^2\theta}{2} - \theta\tan\theta + \int_0^{\frac{\pi}{4}} \frac{d(-\cos\theta)}{\cos\theta} = \theta^2\frac{\sec^2\theta}{2} - \theta\tan\theta + \ln(\sec\theta)$$

代入$\theta = \frac{\pi}{4}$和0　　故等於 $\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}ln2$

Evaluate $\int \frac{xe^x}{\sqrt{e^x - 1}}dx$

[sol]　令$x = ln|u|$　$dx = \frac{1}{u}du$　$u = e^x$

$$\rightarrow \int \frac{ln|u|}{\sqrt{u-1}}du \quad 令u = sec^2\theta \quad du = 2sec^2\theta tan\theta d\theta \quad \theta = \sec^{-1} e^{\frac{x}{2}}$$

$$\rightarrow \int \frac{4ln|sec\theta|}{tan\theta} sec^2\theta tan\theta d\theta = 4\int ln|sec\theta|\, d(tan\theta)$$

$$\rightarrow 4\left(tan\theta ln|sec\theta| - \int tan\theta\, d[ln|sec\theta|]\right)$$

$$\rightarrow 4\left(tan\theta ln|sec\theta| - \int tan^2\theta d\theta\right) = 4\left(tan\theta ln|sec\theta| - \int sec^2\theta - 1 d\theta\right)$$

$$\rightarrow 4tan\theta ln|sec\theta| - 4tan\theta + 4\theta + c = 2x\sqrt{e^x - 1} - 4\sqrt{e^x - 1} + 4\sec^{-1} e^{\frac{x}{2}} + c$$

Evaluate $\int \frac{1+x}{x(1+xe^x)}dx$

[sol]　令$u = 1 + xe^x$　$du = e^x + xe^x dx$

$$\rightarrow \int \frac{e^x + xe^x dx}{xe^x(1+xe^x)} = \int \frac{du}{u(u-1)}$$

將其拆解 $\rightarrow \int \frac{du}{u-1} - \int \frac{du}{u}$

$$= ln\left|\frac{u-1}{u}\right| + c = ln\left|\frac{xe^x}{1+xe^x}\right| + c$$